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Study of Fractional Fourier Series Expansions of Two Types of Matrix Fractional Functions

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Abstract: **In this paper, based on a new multiplication of fractional analytic functions, we obtain the fractional Fourier series expansions of two types of matrix fractional functions. In fact, our results are generalizations of ordinary calculus results.**

Keyword: **New multiplication, fractional analytic functions, fractional Fourier series expansions, matrix fractional functions.**

I. INTRODUCTION

Fractional calculus is a branch of mathematical analysis which deals with the research and applications of integrals and derivatives of arbitrary order. In recent decades, the field of fractional calculus has attracted the interest of researchers in diverse scientific fields such as mechanics, physics, electrical engineering, viscoelasticity, economics, bioengineering, and control theory [1-11]. However, fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [12-16]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on a new multiplication of fractional analytic functions, we find the fractional Fourier series expansions of the following two types of matrix α-fractional functions:

$$\{(pr-pr^3)cos_\alpha(Ax^\alpha)-qr^2cos_\alpha(2Ax^\alpha)+q\}\otimes_\alpha[1-2r^2cos_\alpha(2Ax^\alpha)+r^4]^{\otimes_\alpha(-1)},$$

$$\{(pr+pr^3)sin_\alpha(Ax^\alpha)+qr^2sin_\alpha(2Ax^\alpha)\}\otimes_\alpha[1-2r^2cos_\alpha(2Ax^\alpha)+r^4]^{\otimes_\alpha(-1)},$$

where $0<\alpha\leq 1, p, q, r$ are real numbers, $|r|<1$, and A is a real matrix. In fact, our results are generalizations of classical calculus results.

II. PRELIMINARIES

At first, we introduce the definition of fractional analytic function.

Definition 2.1 ([17])**:** If x, x_0, and a_k are real numbers for all k, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, a new multiplication of fractional analytic functions is introduced.

Definition 2.2 ([18])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha},\tag{1}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\,.\tag{2}$$

Then we define

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\\&=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}.\end{aligned}\tag{3}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}.\end{aligned}\tag{4}$$

Definition 2.3 ([19])**:** If $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n},\tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}.\tag{6}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha(g_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(g_\alpha(x^\alpha)\right)^{\otimes_\alpha n},\tag{7}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha(f_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n}.\tag{8}$$

Definition 2.4 ([20])**:** Let $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n}=f_\alpha(x^\alpha)\otimes_\alpha\cdots\otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)=1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha -1}$.

Definition 2.5: If the complex number $z=p+iq$, where p,q are real numbers, and $i=\sqrt{-1}$. p, the real part of z, is denoted by $\mathrm{Re}(z)$; q the imaginary part of z, is denoted by $\mathrm{Im}(z)$.

Theorem 2.6 (matrix fractional Euler's formula)([21])**:** *If* $0<\alpha\leq 1$, *and* A *is a real matrix, then*

$$E_\alpha(iAx^\alpha)=\cos_\alpha(Ax^\alpha)+i\sin_\alpha(Ax^\alpha).\tag{9}$$

Theorem 2.7 (matrix fractional DeMoivre's formula)([22])**:** *If* $0<\alpha\leq 1$, p *is an integer, and* A *is a real matrix, then*

$$[\cos_\alpha(Ax^\alpha)+i\sin_\alpha(Ax^\alpha)]^{\otimes_\alpha p}=\cos_\alpha(pAx^\alpha)+i\sin_\alpha(pAx^\alpha).\tag{10}$$

III. MAIN RESULTS

In this section, we find the fractional Fourier series expansions of two types of matrix fractional functions. At first, two lemmas are needed.

Lemma 3.1: *If* $0<\alpha\leq 1$, p,q,r *are real numbers, and* A *is a real matrix, then*

$$\mathrm{Re}\left\{[prE_\alpha(iAx^\alpha)+q]\otimes_\alpha\left[1-[rE_\alpha(iAx^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}\right\}$$

$$=\{(pr-pr^3)cos_\alpha(Ax^\alpha)-qr^2cos_\alpha(2Ax^\alpha)+q\}\otimes_\alpha[1-2r^2cos_\alpha(2Ax^\alpha)+r^4]^{\otimes_\alpha(-1)}. \quad (11)$$

$$\mathrm{Im}\left\{[prE_\alpha(iAx^\alpha)+q]\otimes_\alpha\left[1-[rE_\alpha(iAx^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}\right\}$$

$$=\{(pr+pr^3)sin_\alpha(Ax^\alpha)+qr^2sin_\alpha(2Ax^\alpha)\}\otimes_\alpha[1-2r^2cos_\alpha(2Ax^\alpha)+r^4]^{\otimes_\alpha(-1)}. \quad (12)$$

Proof By matrix fractional Euler's formula and matrix fractional DeMoivre's formula,

$$[prE_\alpha(iAx^\alpha)+q]\otimes_\alpha\left[1-[rE_\alpha(iAx^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}$$

$$=[pr[cos_\alpha(Ax^\alpha)+isin_\alpha(Ax^\alpha)]+q]\otimes_\alpha\left[1-r^2[cos_\alpha(2Ax^\alpha)+isin_\alpha(2Ax^\alpha)]\right]^{\otimes_\alpha(-1)}$$

$$=\left[[prcos_\alpha(Ax^\alpha)+q]+iprsin_\alpha(Ax^\alpha)\right]\otimes_\alpha\left[[1-r^2cos_\alpha(2Ax^\alpha)]-ir^2sin_\alpha(2Ax^\alpha)\right]^{\otimes_\alpha(-1)}$$

$$=\left[[prcos_\alpha(Ax^\alpha)+q]+iprsin_\alpha(Ax^\alpha)\right]\otimes_\alpha\left[[1-r^2cos_\alpha(2Ax^\alpha)]+ir^2sin_\alpha(2Ax^\alpha)\right]$$

$$\otimes_\alpha\left[[1-r^2cos_\alpha(2Ax^\alpha)]^{\otimes_\alpha 2}+[r^2sin_\alpha(2Ax^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}$$

$$=\left[[prcos_\alpha(Ax^\alpha)+q]+iprsin_\alpha(Ax^\alpha)\right]\otimes_\alpha\left[[1-r^2cos_\alpha(2Ax^\alpha)]+ir^2sin_\alpha(2Ax^\alpha)\right]$$

$$\otimes_\alpha[1-2r^2cos_\alpha(2Ax^\alpha)+r^4]^{\otimes_\alpha(-1)}. \quad (13)$$

Therefore,

$$\mathrm{Re}\left\{[prE_\alpha(iAx^\alpha)+q]\otimes_\alpha\left[1-[rE_\alpha(iAx^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}\right\}$$

$$=\{[prcos_\alpha(Ax^\alpha)+q]\otimes_\alpha[1-r^2cos_\alpha(2Ax^\alpha)]$$
$$-pr^3sin_\alpha(Ax^\alpha)\otimes_\alpha sin_\alpha(2Ax^\alpha)\}\otimes_\alpha[1-2r^2cos_\alpha(2Ax^\alpha)+r^4]^{\otimes_\alpha(-1)}$$

$$=\{prcos_\alpha(Ax^\alpha)+q-qr^2cos_\alpha(2Ax^\alpha)-pr^3\cos_\alpha(Ax^\alpha)\}\otimes_\alpha[1-2r^2cos_\alpha(2Ax^\alpha)+r^4]^{\otimes_\alpha(-1)}$$

$$=\{(pr-pr^3)cos_\alpha(Ax^\alpha)-qr^2cos_\alpha(2Ax^\alpha)+q\}\otimes_\alpha[1-2r^2cos_\alpha(2Ax^\alpha)+r^4]^{\otimes_\alpha(-1)}.$$

And

$$\mathrm{Im}\left\{[prE_\alpha(iAx^\alpha)+q]\otimes_\alpha\left[1-[rE_\alpha(iAx^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}\right\}$$

$$=\{[prcos_\alpha(Ax^\alpha)+q]\otimes_\alpha r^2sin_\alpha(2Ax^\alpha)$$
$$+prsin_\alpha(Ax^\alpha)\otimes_\alpha[1-r^2cos_\alpha(2Ax^\alpha)]\}\otimes_\alpha[1-2r^2cos_\alpha(2Ax^\alpha)+r^4]^{\otimes_\alpha(-1)}$$

$$=\{prsin_\alpha(Ax^\alpha)+qr^2sin_\alpha(2Ax^\alpha)+pr^3\sin_\alpha(Ax^\alpha)\}\otimes_\alpha[1-2r^2cos_\alpha(2Ax^\alpha)+r^4]^{\otimes_\alpha(-1)}$$

$$=\{(pr+pr^3)sin_\alpha(Ax^\alpha)+qr^2sin_\alpha(2Ax^\alpha)\}\otimes_\alpha[1-2r^2cos_\alpha(2Ax^\alpha)+r^4]^{\otimes_\alpha(-1)}.$$ q.e.d.

Lemma 3.2: *If* $0<\alpha\leq 1$, p,q,r *are real numbers,* $|r|<1$, *and* A *is a real matrix, then*

$$[prE_\alpha(iAx^\alpha)+q]\otimes_\alpha\left[1-[rE_\alpha(iAx^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}$$

$$=p\cdot\sum_{k=0}^{\infty}r^{2k+1}E_\alpha(i(2k+1)Ax^\alpha)+q\cdot\sum_{k=0}^{\infty}r^{2k}E_\alpha(i2kAx^\alpha)\,. \quad (14)$$

Proof $[prE_\alpha(iAx^\alpha)+q]\otimes_\alpha\left[1-[rE_\alpha(iAx^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}$

$$= [prE_\alpha(iAx^\alpha) + q]\otimes_\alpha \sum_{k=0}^{\infty}\left([rE_\alpha(iAx^\alpha)]^{\otimes_\alpha 2}\right)^{\otimes_\alpha k}$$

$$= [prE_\alpha(iAx^\alpha) + q]\otimes_\alpha \sum_{k=0}^{\infty} r^{2k}E_\alpha(i2kAx^\alpha) \quad \text{(by matrix fractional DeMoivre's formula)}$$

$$= p\cdot\sum_{k=0}^{\infty} r^{2k+1}E_\alpha(i(2k+1)Ax^\alpha) + q\cdot\sum_{k=0}^{\infty} r^{2k}E_\alpha(i2kAx^\alpha)\,. \qquad \text{q.e.d.}$$

Theorem 3.3: *If* $0 < \alpha \leq 1$, p, q, r *are real numbers,* $|r|<1$, *and* A *is a real matrix, then*

$$\{(pr - pr^3)cos_\alpha(Ax^\alpha) - qr^2cos_\alpha(2Ax^\alpha) + q\}\otimes_\alpha [1 - 2r^2cos_\alpha(2Ax^\alpha) + r^4]^{\otimes_\alpha(-1)}$$
$$= p\cdot\sum_{k=0}^{\infty} r^{2k+1}cos_\alpha((2k+1)Ax^\alpha) + q\cdot\sum_{k=0}^{\infty} r^{2k}cos_\alpha(2kAx^\alpha)\,. \tag{15}$$

And

$$\{(pr + pr^3)sin_\alpha(Ax^\alpha) + qr^2sin_\alpha(2Ax^\alpha)\}\otimes_\alpha [1 - 2r^2cos_\alpha(2Ax^\alpha) + r^4]^{\otimes_\alpha(-1)}$$
$$= p\cdot\sum_{k=0}^{\infty} r^{2k+1}sin_\alpha((2k+1)Ax^\alpha) + q\cdot\sum_{k=0}^{\infty} r^{2k}sin_\alpha(2kAx^\alpha)\,. \tag{16}$$

Proof By Lemma 3.1 and Lemma 3.2,

$$\{(pr - pr^3)cos_\alpha(Ax^\alpha) - qr^2cos_\alpha(2Ax^\alpha) + q\}\otimes_\alpha [1 - 2r^2cos_\alpha(2Ax^\alpha) + r^4]^{\otimes_\alpha(-1)}$$

$$= \text{Re}\left\{[prE_\alpha(iAx^\alpha) + q]\otimes_\alpha \left[1 - [rE_\alpha(iAx^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}\right\}$$

$$= \text{Re}\{p\cdot\sum_{k=0}^{\infty} r^{2k+1}E_\alpha(i(2k+1)Ax^\alpha) + q\cdot\sum_{k=0}^{\infty} r^{2k}E_\alpha(i2kAx^\alpha)\}$$

$$= p\cdot\sum_{k=0}^{\infty} r^{2k+1}cos_\alpha((2k+1)Ax^\alpha) + q\cdot\sum_{k=0}^{\infty} r^{2k}cos_\alpha(2kAx^\alpha)\,. \quad \text{(by matrix fractional Euler's formula)}$$

Similarly,

$$\{(pr + pr^3)sin_\alpha(Ax^\alpha) + qr^2sin_\alpha(2Ax^\alpha)\}\otimes_\alpha [1 - 2r^2cos_\alpha(2Ax^\alpha) + r^4]^{\otimes_\alpha(-1)}$$

$$= \text{Im}\left\{[prE_\alpha(iAx^\alpha) + q]\otimes_\alpha \left[1 - [rE_\alpha(iAx^\alpha)]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}\right\}$$

$$= \text{Im}\{p\cdot\sum_{k=0}^{\infty} r^{2k+1}E_\alpha(i(2k+1)Ax^\alpha) + q\cdot\sum_{k=0}^{\infty} r^{2k}E_\alpha(i2kAx^\alpha)\}$$

$$= p\cdot\sum_{k=0}^{\infty} r^{2k+1}sin_\alpha((2k+1)Ax^\alpha) + q\cdot\sum_{k=0}^{\infty} r^{2k}sin_\alpha(2kAx^\alpha)\,. \qquad \text{q.e.d.}$$

IV. CONCLUSION

In this paper, based on a new multiplication of fractional analytic functions, we obtain the fractional Fourier series expansions of two types of matrix fractional functions. In fact, our results are generalizations of ordinary calculus results. In the future, we will continue to use the new multiplication of fractional analytic functions to study the problems in fractional differential equations and engineering mathematics.

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